

Partially ample line bundles on toric varieties

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Ample line bundles are a fundamental concept in algebraic geometry, encapsulating the central theme of positivity. A natural extension of the notion of ampleness is that of q -*ampleness*, for nonnegative integers q . Roughly speaking, q -ample line bundles on a variety are those which “kill cohomology in degrees above q ”. Totaro [Tot] showed that q -ample line bundles enjoy many of the good properties of ample line bundles: for instance, the property of q -ampleness is open in families, and also open in the space of line bundles up to numerical equivalence.

A key insight of Mori in the early 1980s is that the convex geometry of the set of ample line bundles on a variety gives important geometric information about the variety. For Fano varieties there is a satisfying description of the set of ample line bundles, given by the Cone Theorem. It remains an open problem to give a good description of the cone of q -ample line bundles on Fano varieties, for general q . For example, Chen–Lazarsfeld asked if the q -ample cone of a Fano variety is the interior of a finite union of rational polyhedral cones, for all q . (This would be the simplest possible description of these cones, since they are known not to be convex in general.)

In this note we give a positive answer to the analogue of Chen–Lazarsfeld’s question in the setting of toric varieties: that is, the q -ample cone of a simplicial projective toric variety in characteristic zero is always the interior of a finite union of polyhedral cones (Theorem 3.3). The proof has two main ingredients: Totaro’s definition of q -*T-ampleness*, which gives a finite characterisation of q -ampleness, and a method of computing cohomology of line bundles on toric varieties in terms of constructible sheaves on the polytope [Br].

In the final section we use the main theorem to prove a characterisation of the q -ample cone of a simplicial projective toric variety in terms of the vanishing of Küronya’s asymptotic cohomological functions.

After completing this paper, we learned that J. C. Ottem [Ott] has obtained the same results independently.

1 q -ample line bundles

In the 1950s Serre gave a cohomological characterisation of ample line bundles: a line bundle is ample if and only if some sufficiently high power of it kills cohomology of any coherent sheaf in degrees above zero. This characterisation suggests the following generalisation of ampleness, introduced by Sommese [Som]. (Note that Sommese’s definition requires that some power of the line bundle be globally generated, but we drop that hypothesis here.)

Definition 1.1. *Let X be a projective variety. A line bundle L on X is called q -ample (for some integer $q \geq 0$) if for any coherent sheaf F on X , there exists a natural number n_0*

(depending on F) such that

$$H^i(X, L^n \otimes F) = 0 \text{ for all } i > q \text{ and } n \geq n_0.$$

Any line bundle on a variety of dimension n is n -ample; by Serre, 0-ample is the same as ample.

At first sight the q -ample condition seems hard to check since it involves tensoring with an arbitrary coherent sheaf. Totaro [Tot, Definition 6.1] introduced the related notion of q -T-*ampleness*, which has the key advantage that it is defined by the vanishing of finitely many cohomology groups:

Definition 1.2. *Let X be a projective variety of dimension n , and fix a Koszul-ample line bundle $\mathcal{O}_X(1)$ on X . A line bundle L on X is called q -T-ample if there exists a natural number N such that*

$$H^{q+1}(X, L^N(-n-1)) = H^{q+2}(X, L^N(-n-2)) = \cdots = H^n(X, L^N(-2n+q)) = 0.$$

For the definition of Koszul-ample see [Tot]; the precise meaning will not play an important role here. For our purposes it will be sufficient to know that:

- i) some sufficiently high power of any ample line bundle is Koszul-ample [Ba],
- ii) the definition of q -T-*ampleness* is independent of the choice of Koszul-ample line bundle [Tot, Corollary 6.2, Theorem 6.3].

The significance of this definition is that the two notions of q -*ampleness* coincide in characteristic zero [Tot, Theorem 6.3]:

Theorem 1.3 (Totaro). *Let X be a projective variety over a field of characteristic zero. Then line bundle L on X is q -ample if and only if it is q -T-ample.*

So in characteristic zero we can check q -*ampleness* by looking at the cohomology of a finite set of line bundles.

Following [Tot, Section 8], we define an \mathbf{R} -divisor to be q -ample if it is numerically equivalent to a sum $cD + A$ with D a q -ample divisor, c a positive rational number, and A an ample \mathbf{R} -divisor. The q -ample cone $\text{Amp}_q(X)$ is the cone of classes of q -ample \mathbf{R} -divisors in $N^1(X)$. (Here $N^1(X)$ denotes the real vector space $(\text{Div}(X)/\equiv) \otimes \mathbf{R}$, where $\text{Div}(X)$ is the group of all Cartier divisors on X , and \equiv denotes numerical equivalence.) We emphasise that $\text{Amp}_q(X)$ is not a convex cone in general: typically the tensor product of two q -ample line bundles is only $2q$ -ample, not q -ample.

We will use Theorem 1.3 to show that the q -ample cone of a simplicial (equivalently, \mathbf{Q} -factorial) projective toric variety in characteristic zero is the interior of a finite union of rational polyhedral cones. Roughly speaking, each vanishing condition in the definition of q -T-ample defines a polyhedral region, and these regions combine to give the q -ample cone.

For later reference we state Totaro's result [Tot, Theorem 8.3] (building on work of Demailly–Peternell–Schneider) that q -*ampleness* is an open condition in the space of line bundles up to numerical equivalence.

Theorem 1.4 (Demailly–Peternell–Schneider, Totaro). *Let X be a projective variety over a field of characteristic zero. Then the q -ample cone $\text{Amp}_q(X)$ is open in $N^1(X)$.*

2 Cohomology of line bundles on toric varieties

In this section we outline the description [Br] of cohomology of line bundles on toric varieties in terms of constructible sheaves on the polytope. We work over an algebraically closed field \mathbf{k} of characteristic zero.

Let $X = X(\Delta)$ be a simplicial projective n -dimensional toric variety, corresponding to some complete fan Δ in a lattice $N \cong \mathbf{Z}^n$. We denote the primitive generators of the rays of Δ by $\{v_i \in N \mid i \in I\}$. There is a one-to-one correspondence between prime torus-invariant divisors and rays of Δ [Fu, Chapter 3]. We denote these divisors by $\{E_i \mid i \in I\}$ and the free group generated by them by \mathbf{Z}^I . The dual space \mathbf{Z}_I is generated by the dual basis $\{e_i \mid i \in I\}$.

Let $M := \text{Hom}(N, \mathbf{Z}) \cong \mathbf{Z}^n$ be the dual lattice to N , with pairing $\langle \cdot, \cdot \rangle$. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \text{Div}_T(X) & \longrightarrow & \text{Pic}(X) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & \mathbf{Z}^I & \xrightarrow{[-]} & \text{Cl}(X) \longrightarrow 0 \end{array} \quad (1)$$

where $\text{Div}_T(X)$ is the group of torus-invariant Cartier divisors, and $\text{Cl}(X)$ is the class group. Applying the functor $-\otimes_{\mathbf{Z}} \mathbf{R}$ and using the fact that simplicial toric varieties are \mathbf{Q} -factorial, we obtain the following exact sequence

$$0 \longrightarrow M_{\mathbf{R}} \longrightarrow \mathbf{R}^I \xrightarrow{[-]} N^1(X) \longrightarrow 0 \quad (2)$$

where $N^1(X)$ denotes $\text{Pic}(X) \otimes \mathbf{R}$.

We denote the polytope of the toric variety X by P_X . To fix notation, let us recall how this polytope is constructed. For an ample divisor $A = \sum_{i \in I} a_i E_i$ on X , there is an associated n -dimensional polytope in $M_{\mathbf{R}}$ given by

$$P_A := \{u \in M_{\mathbf{R}} \mid \langle u, v_i \rangle \geq -a_i \ \forall i \in I\}.$$

This has $(n-1)$ -dimensional faces

$$F_i := \{u \in M_{\mathbf{R}} \mid \langle u, v_i \rangle = -a_i \text{ and } \langle u, v_j \rangle \geq -a_j \ \forall j \in I \setminus \{i\}\}$$

indexed by the rays of Δ . The combinatorial structure of this polytope is independent of the choice of ample divisor A [Fu, p. 72], so we define P_X to be the polytope P_A for any ample divisor A .

If D is any torus-invariant Cartier divisor on X , its cohomology $H^p(X, \mathcal{O}(D))$ splits into a direct sum of weight spaces indexed by the lattice M [Fu, Section 3.5, Proposition]:

$$H^p(X, \mathcal{O}(D)) \cong \bigoplus_{m \in M} H^p(X, \mathcal{O}(D))_m. \quad (3)$$

There are several known descriptions of these weight spaces, for example Demazure's description [Dem] in terms of local cohomology groups. Here we will use the description from [Br] in terms of cohomology of a constructible sheaf on the polytope P_X . More precisely, if

$D = \sum_{i \in I} d_i E_i$ is the torus-invariant divisor, then for each $m \in M$ we define a closed subset of P_X by

$$Z(m, D) := \bigcup_{\{i \in I \mid \langle m, v_i \rangle < -d_i\}} F_i = \bigcup_{\{i \in I \mid \langle m+D, e_i \rangle < 0\}} F_i,$$

a union of closed maximal dimensional faces of P_X , and denote by $W(m, D) := P_X \setminus Z(m, D)$ the complementary open subset of P_X .

Let $j : W(m, D) \hookrightarrow P_X$ be the inclusion of $W(m, D)$ into P_X and \mathbf{k}_W be the constant sheaf on W with values in the ground field. Then [Br, Theorem 1.1] states that for all $p \geq 0$ there are canonical isomorphisms

$$H^p(X, \mathcal{O}(D))_m \cong H^p(P_X, j! \mathbf{k}_{W(m, D)}). \quad (4)$$

This leads to a practical method of computation as follows. There is a short exact sequence of constructible sheaves

$$0 \longrightarrow j! j^* \mathbf{k}_{P_X} \longrightarrow \mathbf{k}_{P_X} \longrightarrow i_* i^* \mathbf{k}_{P_X} \longrightarrow 0$$

where i is the inclusion of $Z(m, D)$ into P_X , which induces a long exact sequence of cohomology

$$\dots \longrightarrow H^{p-1}(Z(m, D), \mathbf{k}) \longrightarrow H^p(P_X, j! \mathbf{k}_{W(m, D)}) \longrightarrow H^p(P_X, \mathbf{k}) \longrightarrow \dots$$

Using this together with Equation (4) and observing that P_X is contractible, we conclude that for all $p \geq 0$ we have

$$H^p(X, \mathcal{O}(D))_m = \tilde{H}^{p-1}(Z(m, D), \mathbf{k}) \quad (5)$$

where \tilde{H} denotes reduced cohomology. (We use the convention that $\tilde{H}^{-1}(Z, \mathbf{k}) = 0$ except when $Z = \emptyset$, and $\tilde{H}^{-1}(\emptyset, \mathbf{k}) = \mathbf{k}$.)

In order to compute cohomology groups it is therefore important to understand the sets $Z(m, D)$. For each subset $\alpha \subseteq I$, we define the set

$$Z_\alpha := \bigcup_{i \in \alpha} F_i$$

and note that each $Z(m, D)$ is of this form. Given $\alpha \subseteq I$ we define a corresponding orthant in \mathbf{R}^I as follows

$$O_\alpha := \{D = \sum_{i \in I} d_i E_i \in \mathbf{R}^I \mid d_i < 0 \ \forall i \in \alpha, \ d_i \geq 0 \ \forall i \in I \setminus \alpha\}.$$

With these definitions the following lemma is immediate:

Lemma 2.1. $Z(m, D) = Z_\alpha$ if and only if $D + m \in O_\alpha$.

3 The q -ample cone of a toric variety

In this section we will prove our main theorem. First we collect a few basic facts about rational cones that we will need for the proof.

A subset of a vector space is a (closed, not necessarily strongly) convex polyhedral cone if and only if it is the intersection of a finite number of closed half-spaces. We will refer to the intersection of a finite number of half-spaces, some of which are open, as a partially open convex polyhedral cone.

Lemma 3.1. *The image $[O_\alpha]$ of an orthant O_α in $N^1(X)$ is an intersection of a finite number of rational half-spaces (some of which may be open). Its closure $\overline{[O_\alpha]}$ is a convex rational polyhedral cone of top dimension. \square*

Lemma 3.2. *Let C_1, \dots, C_t be convex rational polyhedral cones (some of which may be partially open). The complement of their union is also a finite union of convex rational polyhedral cones (some of which may be partially open).*

Proof. For a single cone C_1 , we write this as an intersection of half spaces. The set of (rational) hyperplanes bounding these half spaces split the complement of C_1 up into a finite number of convex rational polyhedral cones (some of which are partially open). The lemma then follows for a finite union $\bigcup_{j=1}^t C_j$ using De Morgan's law and the fact that the intersection of two rational polyhedral cones is again a rational polyhedral cone. \square

Now we can prove our main theorem.

Theorem 3.3. *The q -ample cone of a simplicial (equivalently, \mathbf{Q} -factorial) projective toric variety X over an algebraically closed field of characteristic zero is the interior of a union of finitely many rational polyhedral cones.*

Proof. We denote by n the dimension of X and let $\mathcal{L} := \mathcal{O}(A) \in \text{Pic}(X)$ be a Koszul-ample line bundle. We start by proving that the set

$$S = \{ \mathcal{O}(D) \in \text{Pic}(X) \mid H^i(X, \mathcal{O}(kD + (q - i - n)A)) = 0, \forall i > q, \forall k \gg 0 \}.$$

is the set of classes of all q -ample Cartier divisors on X . In one direction, the elements of S satisfy a condition which obviously implies q - T -ampleness, and are therefore classes of q -ample Cartier divisors by Theorem 1.3.

In the other direction, suppose that D is a q -ample Cartier divisor. Then for any given coherent sheaf E on X , there exists a positive integer N (depending on E) such that $H^i(kD + E) = 0$ for all $k \geq N$, in all degrees $i > q$. Applying this with E chosen to be $(-n-1)A, (-n-2)A, \dots, (-2n+q)A$, we get natural numbers N_1, N_2, \dots, N_{n-q} such that for each j , and each $k \geq N_j$, we have $H^i(kD + (q - j - n)A) = 0$ for all $i > q$. Now set $N = \max \{N_j\}$: then for each $j = q+1, \dots, n$ and each $k \geq N$ we have $H^i(kD + (q - j - n)A) = 0$ for all $i > q$. In particular, choosing $i = j$, we see that the class of D lies in S .

Therefore, the q -ample cone $\text{Amp}_q(X)$ is the cone of \mathbf{R} -divisor classes of the form $[cD + A']$ where $[D] \in S$, c is a positive rational number and A' is an ample \mathbf{R} -divisor. It remains to prove that this is the interior of a finite union of rational polyhedral cones. Using the weight space decomposition (3) and Equation (5), we can describe the set S defined above as

$$S = \left\{ \mathcal{O}(D) \in \text{Pic}(X) \mid \tilde{H}^j(Z(m, kD + (q - j - n - 1)A), \mathbf{k}) = 0, \forall m \in M, \forall j \geq q, \forall k \gg 0 \right\}.$$

Let $J_i := \{ \alpha \subseteq I \mid \tilde{H}^i(Z_\alpha, \mathbf{k}) \neq 0 \}$ be the set indexing orthants which would contribute non-zero terms to the i -th cohomology, and let

$$O_{J_i} := \bigcup_{\alpha \in J_i} O_\alpha$$

be the union of these orthants. Then using Lemma 2.1 we have

$$S = \{ \mathcal{O}(D) \in \text{Pic}(X) \mid (kD + (q - i - n - 1)A + M) \cap O_{J_i} = \emptyset, \forall i \geq q, \forall k \gg 0 \}.$$

Furthermore, using the short exact sequence (1) it is clear that for any divisor E and subset $U \subseteq \mathbf{Z}^I$ we have

$$(E + M) \cap U = \emptyset \text{ if and only if } [E] \notin [U]$$

and so

$$S = \{\mathcal{O}(D) \in \text{Pic}(X) \mid [kD + (q - i - n - 1)A] \notin [O_{J_i}], \forall i \geq q, \forall k \gg 0\}. \quad (6)$$

Let

$$K := \bigcup_{i \geq q} [O_{J_i}]$$

which, by Lemma 3.1, is a union of (partially open) rational polyhedral cones in $N^1(X)$ and denote by K^c its complement. By Lemma 3.2 the set K^c is also a union of (partially open) rational polyhedral cones in $N^1(X)$. We will show that $\text{Amp}_q = (K^c)^\circ$, which proves the theorem.

First we will show that Amp_q contains the cone $(K^c)^\circ$. Choose any point $[D']$ in $(K^c)^\circ$. Since this set is open, we can write $[D']$ in the form $[D + A']$ where $[D]$ is a rational point in $(K^c)^\circ$ and A' is some ample \mathbf{R} -divisor. There exists $\varepsilon > 0$ such that the ε -ball $B_\varepsilon([D])$ around $[D]$ is contained in K^c . Since K^c is closed under multiplication by a positive scalar, the cone Σ generated by $B_\varepsilon([D])$ is also contained in K^c , and since $B_\varepsilon([D])$ is convex the cone Σ is closed under addition. Let

$$k_0 > \max_{q \leq i < n} \|[D + (q - i - n - 1)A]\|/\varepsilon$$

so for every $q \leq i < n$ we have

$$[D] + \frac{1}{k_0}[D + (q - i - n - 1)A] \in \Sigma$$

and therefore

$$(k_0 + 1)[D] + (q - i - n - 1)[A] \in \Sigma.$$

Then, since Σ is closed under addition, for $k > k_0$ we have

$$\begin{aligned} [kD + (q - i - n - 1)A] &= (k - (k_0 + 1))[D] + ((k_0 + 1)[D] + (q - i - n - 1)[A]) \\ &\in \Sigma \subset K^c. \end{aligned}$$

In particular, $[kD + (q - i - n - 1)A] \notin [O_{J_i}]$ for all $i \geq q$ and for all $k \gg 0$. Therefore D is q -ample and it follows that $(K^c)^\circ \subseteq \text{Amp}_q$.

Next we will show that Amp_q is contained in the cone $\overline{K^c}$, the closure of the complement of K . To see this, suppose for the sake of contradiction that $\text{Amp}_q \cap K^\circ \neq \emptyset$. Since Amp_q and K° are both open and the rational points are dense, there is a rational point $[D] \in \text{Amp}_q \cap K^\circ$. Choose $\varepsilon_0 > 0$ such that $[D] + \varepsilon[A] \in K^\circ$ for all ε with $|\varepsilon| < \varepsilon_0$. In fact since K is by definition a finite union of convex sets, for all ε sufficiently close to zero, the point $[D] + \varepsilon[A]$ must lie in a given one of these sets, so there exists $i \geq q$ and $\tilde{\varepsilon}_0 > 0$ such that $[D] + \varepsilon[A] \in [O_{J_i}]$ for all ε with $0 < |\varepsilon| < \tilde{\varepsilon}_0$. Therefore for all $k \gg 0$, $[D + \frac{(q-i-n-1)}{k}A] \in [O_{J_i}]$ and, multiplying through by k , it follows that $[D] \notin \text{Amp}_q$. This is a contradiction, so we conclude that $\text{Amp}_q \subseteq \overline{K^c}$.

Putting the last two paragraphs together, we have

$$(K^c)^\circ \subseteq \text{Amp}_q \subseteq \overline{K^c}.$$

Using Theorem 1.4 which says that Amp_q is open in $N^1(X)$, this gives

$$\overline{K}^c = (K^c)^\circ \subseteq \text{Amp}_q \subseteq \overline{K}^c = \overline{K}^{\circ c}.$$

Because K is the union of top dimensional convex cones, every point in the closure is the limit point of a sequence of points in the interior (using the convexity and the fact that each cone has non-empty interior) so $\overline{K} = \overline{K}^\circ$. Therefore we obtain the following descriptions of Amp_q :

$$\text{Amp}_q = \overline{K}^c = (K^c)^\circ = \overline{K}^{\circ c} = \overline{K}^{\circ c}. \quad (7)$$

The theorem follows since \overline{K}^c is also a union of finitely many rational polyhedral cones by Lemma 3.2. \square

4 Example

To illustrate the theorem, we will calculate the q -ample cone of the blowup of \mathbf{P}^n in a single point, for all $n \geq 2$ and all interesting values of q .

Let X be the blowup of \mathbf{P}^n in a point, which we may choose to be the intersection of n of the $n+1$ torus-invariant divisors in \mathbf{P}^n . Then X has $n+2$ torus-invariant divisors and $\text{Pic}(X)$ has two generators given by H , the pullback to X of the hyperplane class on \mathbf{P}^n , and E , the class of the exceptional divisor. The short exact sequence from diagram (1) is

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix} \mathbf{I}_{n \times n} \\ -1 \dots -1 \\ 1 \dots 1 \end{pmatrix}} \mathbf{Z}^{n+2} \xrightarrow{\begin{pmatrix} 1 & \dots & 1 & \mathbf{I}_{2 \times 2} \\ -1 & \dots & -1 & \end{pmatrix}} \mathbf{Z}^2 \longrightarrow 0$$

The polytope P_X is obtained by “chopping off” one vertex of the n -simplex. We label the top-dimensional faces of this polytope by $F_1, \dots, F_{n+1}, F_{n+2}$, ordered so that F_{n+1} and F_{n+2} are the two disjoint faces.

We need to understand the reduced homology groups of the subspaces $Z(m, D) \subset P_X$ defined in Section 2.

Lemma 4.1. *Let Y be a union of closed top-dimensional faces of the polytope P_X . Then its reduced homology groups are*

$$\tilde{H}_k(Y) = \begin{cases} \mathbf{k} & \text{if } Y = \partial P_X, k = n-1 \\ \mathbf{k} & \text{if } Y = \partial P_X \setminus \{F_{n+1} \cup F_{n+2}\}, k = n-2 \\ \mathbf{k} & \text{if } Y = F_{n+1} \cup F_{n+2}, k = 0 \\ \mathbf{k} & \text{if } Y = \emptyset, k = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof comes from considering the long exact sequence of reduced homology associated to a sequence $A \hookrightarrow B \rightarrow B/A$, where A is a (reasonable) closed subspace of a topological space B . Applying this with $B = Y$, a union of top-dimensional faces of P_X , and $A = Y \cap F_{n+2}$ we reduce the problem to calculating the reduced homology of either a union of faces of a simplex or the disjoint union of a point with a union of faces of a simplex. Using the fact that the union of any proper subset of faces of a simplex has no reduced homology, the result follows. \square

This lemma immediately identifies the index sets J_α which give nonzero contributions to cohomology of a line bundle, as described in Section 2. We have

$$\begin{aligned} J_0 &= \left\{ \alpha \subseteq I = \{1, \dots, n+2\} \mid \tilde{H}^0(Z_\alpha, \mathbf{k}) \neq 0 \right\} = \{\{n+1, n+2\}\} \\ J_1 &= \dots = J_{n-3} = \emptyset \\ J_{n-2} &= \left\{ \alpha \subseteq I \mid \tilde{H}^{n-2}(Z_\alpha, \mathbf{k}) \neq 0 \right\} = \{\{1, \dots, n\}\} \\ J_{n-1} &= \left\{ \alpha \subseteq I \mid \tilde{H}^{n-1}(Z_\alpha, \mathbf{k}) \neq 0 \right\} = \{\{1, \dots, n+2\}\}. \end{aligned}$$

The corresponding orthants in $\mathbf{Z}^I \cong \mathbf{Z}^{n+2}$ are then

$$\begin{aligned} O_{J_0} &= \{(d_1, \dots, d_{n+2}) \in \mathbf{Z}^{n+2} \mid d_{n+1} < 0, d_{n+2} < 0, d_i \geq 0 \text{ for all } i = 1, \dots, n\} \\ O_{J_1} &= \dots = O_{J_{n-3}} = \emptyset \\ O_{J_{n-2}} &= \{(d_1, \dots, d_{n+2}) \in \mathbf{Z}^{n+2} \mid d_{n+1} \geq 0, d_{n+2} \geq 0, d_i < 0 \text{ for all } i = 1, \dots, n\} \\ O_{J_{n-1}} &= \{(d_1, \dots, d_{n+2}) \in \mathbf{Z}^{n+2} \mid d_i < 0 \text{ for all } i = 1, \dots, n+2\} \end{aligned}$$

The images of these orthants under the map $\mathbf{Z}^I \rightarrow \text{Pic}(X)$ then look as in Figure 1.

In the proof of Theorem 3.3 we showed that the q -ample cone is given by $\text{Amp}_q = (K^c)^o$, where $K = \bigcup_{i \geq q} [O_{J_i}]$. So we can compute the q -ample cone for each q by forming appropriate unions of the regions $\overline{[O_{J_i}]}$ and taking the complement. The results are shown in Figure 2.

We remark that Totaro [Tot, Theorem 9.1] showed that Amp_{n-1} is the complement of the negative of the closed effective cone, for any projective variety in characteristic zero. For X the blowup of \mathbf{P}^n in a point, it is easy to show that the effective cone is closed, spanned by E and $H - E$, and the complement of the negative of that cone is indeed the cone Amp_{n-1} in Figure 2.

5 Asymptotic cohomological functions and q -ampleness

Totaro [Tot, Section 10] asked whether the q -ample cone could be characterised by the vanishing of Küronya's asymptotic cohomological functions in degrees above q . (For the ample cone, this was proved by de Fernex–Küronya–Lazarsfeld [dFKL].)

Theorem 5.1. *Let D be a Cartier divisor on a projective toric variety. Then D is q -ample if and only if \hat{h}^i vanishes identically in an open neighbourhood of $[D]$ in $N^1(X)$ for all $i > q$.*

Proof. First assume D is q -ample. Küronya [Kür] showed that the functions \hat{h}^i are homogeneous of degree equal to the dimension of X , and continuous in the Euclidean topology on $N^1(X)$, so to prove the claim it suffices to show that \hat{h}^i vanishes for all Cartier divisors spanning rays which pass sufficiently close to $[D]$. By openness of the q -ample cone, there exists an open neighbourhood U of $[D]$ in $N^1(X)$ which also lies in the q -ample cone. Any Cartier divisor D' lying on a ray which intersects U is q -ample, so applying Definition 1.1 with $F = \mathcal{O}_X$ we get $H^i(X, \mathcal{O}(nD')) = 0$ for n sufficiently large and for $i > q$. Therefore $\hat{h}^i(D') = 0$, as required.

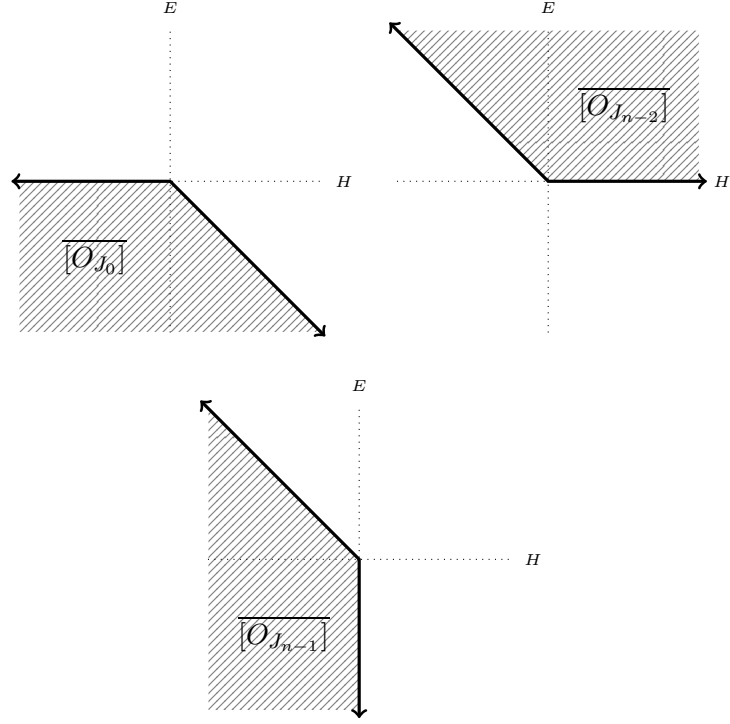


Figure 1: Orthants of cohomology vanishing

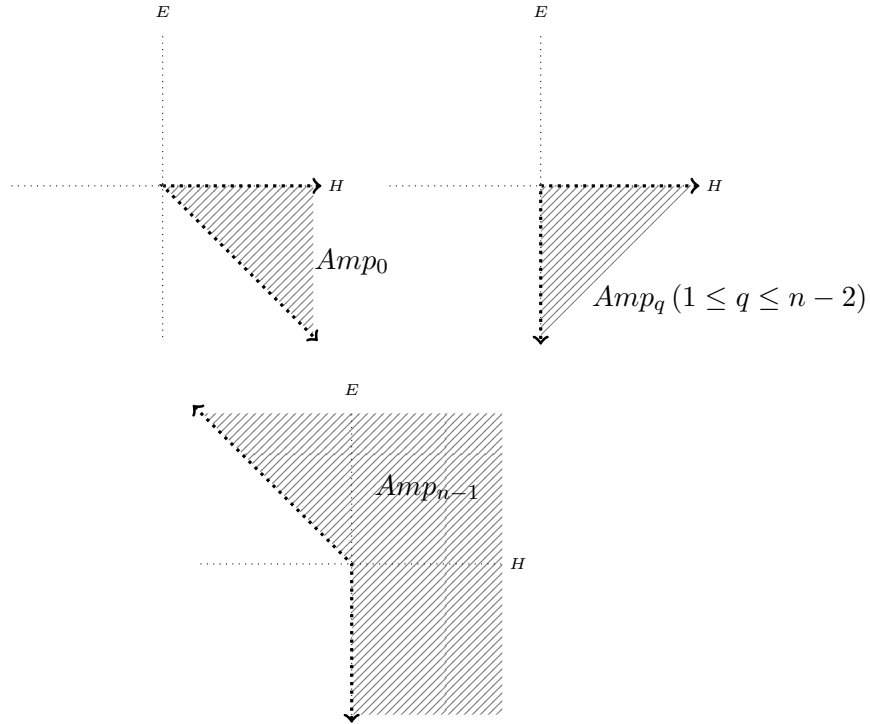


Figure 2: The q -ample cones

To prove the converse, suppose D is not q -ample. Using the description of the q -ample cone in the proof of Theorem 3.3, we have $[D] \in \overline{K}$. Therefore given any open neighbourhood of $[D]$ in $N^1(X)$, it intersects the interior of K . In fact it follows from the definition of K that there exists some $i \geq q$ and $\alpha \in J_i$ such that the neighbourhood intersects the interior of $[O_\alpha]$. It will therefore be sufficient to show that for any Cartier divisor D' with $[D'] \in [O_\alpha]^\circ$ the function $\widehat{h}^{i+1}(D') \neq 0$. We choose a representative divisor D' of the class $[D']$ which lies in the interior of the orthant O_α . If U denotes the closed unit hypercube in M we fix a large enough integer k such that $D' + \frac{1}{k}U \subset O_\alpha$. It follows that for each positive integer j we have $jkD' + jU \subset O_\alpha$ and so for each $m \in jU$ the m th piece of the cohomology $H^{i+1}(X, \mathcal{O}(jkD'))_m \neq 0$. Therefore $h^{i+1}(X, \mathcal{O}(jkD')) \geq j^n$ and it follows that $\widehat{h}^{i+1}(D') \neq 0$. \square

References

- [Ba] J. Backelin. On the rates of growth of the homologies of Veronese subrings. *Algebra, algebraic topology and their interactions* (Stockholm, 1983), 79-100. LNM 1183, Springer (1986).
- [Br] N. Broomhead. Cohomology of line bundles on a toric variety and constructible sheaves on its polytope. arXiv:math/0611469
- [Dem] M. Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup.* **3** (1970), 507-588.
- [dFKL] T. de Fernex, A. Küronya, R. Lazarsfeld. Higher cohomology of divisors on a projective variety. *Math. Ann.* **337** (2007), no. 2, 443-455.
- [Fu] W. Fulton. *Introduction to toric varieties*. Annals of Mathematics Studies, 131. Princeton (1993).
- [Iv] B. Iversen. *Cohomology of sheaves*. Springer (1986).
- [Kür] A. Küronya. Asymptotic cohomological functions on projective varieties. *Amer. J. Math.* **128** (2006), 1475-1519.
- [Ott] J.C. Ottem. Ph.D. thesis, University of Cambridge.
- [Som] A. Sommese. Submanifolds of Abelian varieties. *Math. Ann.* **233** (1978), 229-256.
- [Tot] B. Totaro. Line bundles with partially vanishing cohomology. *J. Eur. Math. Soc.*, to appear.